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KELLER-BOX SCHEME TO MIXED CONVECTION FLOW OVER A SOLID SPHERE WITH THE EFFECT OF MHD

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ABSTRAK

Konveksi campuran merupakan gabungan dari konveksi bebas yang disebabkan oleh gaya apung akibat perbedaan densitas dan konveksi paksa akibat gaya luar yang meningkatkan laju pertukaran panas. Artinya, pada konveksi bebas, pengaruh gaya luar juga signifikan selain gaya apung. Dalam penelitian ini jenis fluida yang memiliki efek viskoelastik adalah non-Newtonian. Cairan viskoelastik yang melewati permukaan bola membentuk lapisan tipis, yang karena viskositasnya yang dominan disebut dengan lapisan batas. Lapisan pembatas yang diperoleh dianalisis dengan ketebalan lapisan batas-y di dekat titik stagnasi bawah, kemudian diperoleh persamaan dimensi lapisan batas, kontinuitas, momentum, dan persamaan energi. Persamaan lapisan batas dimensi ini kemudian diubah menjadi persamaan lapisan batas non dimensi dengan menggunakan variabel non dimensi. Selanjutnya persamaan lapisan batas non dimensional ditransformasikan menjadi persamaan differensial biasa dengan menggunakan fungsi stream, sehingga diperoleh persamaan lapisan batas yang tidak serupa. Persamaan lapisan batas tidak serupa diselesaikan secara numerik dengan menggunakan metode beda hingga dari Keller-Box. Hasil diskritisasi tidak linier dan harus dilinierisasi menggunakan teknik linierisasi newton. Solusi numerik menganalisis pengaruh parameter bilangan Prandtl, viskoelastik, konveksi campuran, dan maghnetohidrodinamik terhadap profil kecepatan, profil suhu, dan suhu dinding.

Kata kunci: teori lapisan batas, aliran konveksi campuran, Navier-Stokes, fluida viskoelastik.

ABSTRACT

Mixed convection is the combination of a free convection caused by the buoyancy forces due to the different density and a forced convection due to external forces that increase the heat exchange rate. This means that, in free convection, the effect of external forces is significant besides buoyancy forces. In this study the fluid type with viscoelastic effect is non-Newtonian. The viscoelastic fluids that pass over a surface of a sphere form a thin layer, which due to their dominant viscosity is called by the border layer. The obtained limiting layer is analyzed with the thickness of the boundary layer-y near the lower stagnating point, then obtained dimensional boundary layer equations, continuity, momentum, and energy equations. These dimensional boundary layer equations are then transformed into non-dimensional boundary layer equations by using non-dimensional variables. Further, the non-dimensional boundary layer equations are transformed into ordinary differential equations by using stream function, so that obtained the non-similar boundary layer equations. These non-similar boundary layer equations are solved numerically by using finite difference method of Keller-Box. The discretization results are non-linear and it should be linearized using newton linearization technique. The numerical solutions are analyzed the effect of Prandtl number, viscoelastic, mixed convection, and MHD parameters towards velocity profile, temperature profile, and wall temperature.

Keywords: boundary layer theory, mixed convection flow, Navier-Stokes, viscoelastic fluid.

INTRODUCTION

The boundary layer problems of mixed convection flow over a sphere are fundamental theory and have been applied widely in engineering applications. Many researchers have investigated these problems in different geometries such as flat plate, cone, and cylinder with type of fluids Newtonian or non-Newtonian. Boundary layer on fluid is a layer near surface of medium so the effect of viscosity and velocity profile to be significant because of shear stress at the wall (Sleigh and Andrew, 2001). In this research, the mixed convection flow that is the combination of free convection flow and forced convection flow is analyzed (Kreith and Frank, 1994). The researches of mixed convection over a sphere have been studied by several researchers such as Amin et al (2002) studied mixed convection flow over a surface of sphere in steady state and incompressible with the constant temperature. Further, the numerical solutions were solved by the Keller-Box method. Nazar et al (2010) studied mixed convection flow over a sphere with Newtonian heating. Heat transfer of Newtonian heating was proportional to local surface temperature. Salleh and Ibrahim (2002) studied mixed convection flow over a sphere at lower stagnation point with Newtonian heating. Temperature profile and velocity profile were analyzed based on mixed convection parameter and Prandtl number. Kasim (2014) studied mixed convection flow of viscoelastic fluid over a sphere in steady-state and incompressible that was solved numerically by the Keller-Box method. A similar mathematical model with the current paper was studied by Ghani, et al (2014) and (2015) for case of mixed convection flow, where Crank-Nicolson and iterative method were employed to establish the numerical results. Moreover, the numerical solution for case of free convection flow over a sphere was investigated by Rumite, et al (2015). Based on the previous studies, we investigate the velocity and temperature profile to mixed convection flow of viscoelastic fluid over a surface of sphere with the effect of MHD in steady state and incompressible. These non-similar equations are solved numerically using the finite difference method of Keller-Box with newton linearization technique to solve non-linear ordinary differential equations. In this research, it is only investigated laminar flow of viscoelastic fluid over a sphere surface. This means that the velocity of fluid is small because of the viscoelastic effect that is shown by the Reynolds number $R_e < 500$ (Widodo, 2012).

MATHEMATICAL MODELING

Consider steady-state and incompressible two-dimensional mixed convection flow of viscoelastic fluid flow over a sphere with the effect of MHD where a is radius of sphere. The physical model of this research is illustrated as follows.

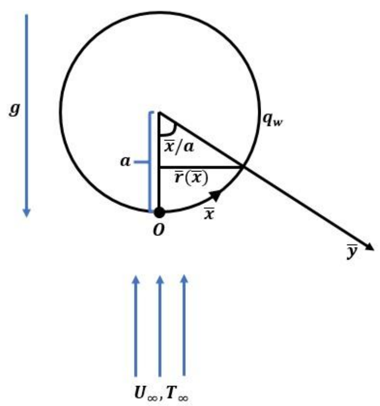

Figure 1. Physical Model of Mixed Convection of Viscoelastic Fluid Over a Solid Sphere

Figure 1 gives illustration of the physical model and coordinate system to mixed convection flow of viscoelastic fluid over a surface of solid sphere. Heat flux q_w on the surface of sphere can affect the temperature around the surface of sphere becomes increased because of the friction between the viscoelastic of fluid and surface of sphere. T_∞ denotes the temperature of fluid past a sphere. Since the different density of temperature between fluid and surface of sphere, then the fluid goes upward where the gravity g is considered in the current paper. Moreover, $\bar{r}(\bar{x})$ and a are radial distance and radius of sphere respectively, where the radial distance $\bar{r}(\bar{x})$ is more detail written in APPENDIX. Based on the Boussines q and boundary layer approximations, then one has the basic equations of continuity, momentum, and energy equations that have been studied by Widodo (2013) and Kasim (2014).

$$\frac{\partial}{\partial \bar{x}}(\bar{r}\bar{u}) + \frac{\partial}{\partial \bar{y}}(\bar{r}\bar{v}) = 0 \tag{1}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \overline{u_e}\frac{\partial \overline{u_e}}{\partial \bar{x}} + \bar{v}\left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\right] - \frac{k_0}{\rho}\left[\bar{u}\left(\frac{\partial^3 \bar{u}}{\partial \bar{x}^3 \bar{y}^2}\right) + \bar{v}\frac{\partial^3 \bar{u}}{\partial \bar{y}^3} + \frac{\partial \bar{u}}{\partial \bar{x}}\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)\right] +$$

$$\frac{k_0}{\rho}\left[\frac{\partial \bar{u}}{\partial \bar{y}}\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}\partial \bar{x}}\right)\right] - \ g\beta(\bar{T} - \bar{T}_\infty)\sin\left(\frac{\bar{x}}{\bar{a}}\right) - \frac{1}{\rho}\sigma(\bar{u} - \overline{u_e})B_0^2 \tag{2}$$

$$\left(\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}}\right) = \alpha\frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \tag{3}$$

with the boundary conditions.

$$\bar{u} = \bar{v} = 0, \frac{\partial \bar{T}}{\partial \bar{y}} = -\frac{q_w}{k}\ at\ \bar{y} = 0$$

$$\bar{u} = \overline{u_e}(x), \frac{\partial \bar{u}}{\partial \bar{y}} = 0, T = T_\infty\ at\ \bar{y} \to \infty \tag{4}$$

where $u_e(x)$ is velocity of local free flow at the outside of boundary layer that is defined by $u_e(x) = \frac{3}{2}U_\infty \sin\left(\frac{x}{a}\right)$. The non-dimensional variables are then given as follows.

$$x = \frac{\bar{x}}{a}, y = R_e^{\frac{1}{2}}\left(\frac{\bar{y}}{a}\right), r(x) = \frac{\bar{r}(\bar{x})}{a}, u = \frac{\bar{u}}{U_\infty},$$

$$v = R_e^{\frac{1}{2}}\left(\frac{\bar{v}}{U_\infty}\right), \theta = \frac{R_e^{\frac{1}{2}}(T - T_\infty)k}{q_w a}, u_e(x) = \frac{\overline{u_e}(\bar{x})}{U_\infty} \tag{5}$$

By substituting (5) into (1) to (3), then one has the non-dimensional equations.

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(rv) = 0 \quad (6)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{\partial u_e}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \lambda\theta\sin(x) - K\left[v\frac{\partial^3 u}{\partial y^3} + u\frac{\partial^3 u}{\partial x \partial y^2}\right]$$

$$+K\left[\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}\right] - \; M(u - u_e) \quad (7)$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{P_r}\frac{\partial^2 \theta}{\partial y} \quad (8)$$

where K and λ are non-dimensional parameters of viscoelastic and mixed convection respectively that are defined as $K = \frac{k_0}{\rho}\left(\frac{U_\infty}{a\nu}\right)$ and $\lambda = \frac{G_r}{R_e^{\frac{5}{2}}}$ respectively with the following boundary conditions.

$$u = v = 0, \theta' = -1 \; at \; y = 0$$

$$u_e = \frac{3}{2}\sin(x), \frac{\partial u}{\partial y} = 0, \theta = 0 \; at \; y \to \infty \quad (9)$$

Furthermore, it follows from(9), then (6) to (8) can be solved using the stream function in (10).

$$\psi = xr(x)f(x,y), \theta = \theta(x,y) \quad (10)$$

where ψ is defined as

$$u = \frac{1}{r}\frac{\partial \psi}{\partial y} \text{ and } v = -\frac{1}{r}\frac{\partial \psi}{\partial x} \quad (11)$$

Based on (11), then (6) to (8) are written as the non-similar equations.

$$\left(\frac{\partial^3 f}{\partial y^3}\right) + \left(1 + x\frac{\cos(x)}{\sin(x)}\right)f\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial f}{\partial y}\right)^2 + \frac{9}{4} + \lambda\theta\frac{\sin(x)}{x} - 2K\left[\frac{\partial f}{\partial y}\frac{\partial^3 f}{\partial y^3}\right]$$

$$+K\left[\left(1 + x\frac{\cos(x)}{\sin(x)}\right)\left(f\frac{\partial^4 f}{\partial y^4} + \left(\frac{\partial^2 f}{\partial y^2}\right)^2\right)\right] = x\left(\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y^2}\frac{\partial f}{\partial x}\right) - M\frac{\partial f}{\partial y}$$

$$+\frac{3}{2}M\frac{\sin x}{x} + Kx\left[\frac{\partial^3 f}{\partial y^3}\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^4 f}{\partial y^4}\frac{\partial f}{\partial x} - x\frac{\partial^2 f}{\partial y^2}\frac{\partial^3 f}{\partial x \partial y^2} + \frac{\partial f}{\partial y}\frac{\partial^4 f}{\partial x \partial y^3}\right] \quad (12)$$

$$x\left(\frac{\partial f}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial \theta}{\partial y}\right) = \frac{1}{P_r}\frac{\partial^2 \theta}{\partial y} + \left(1 + x\frac{\cos(x)}{\sin(x)}\right)f\frac{\partial \theta}{\partial y} \quad (13)$$

with the boundary conditions.

$$f = 0, \frac{\partial f}{\partial y} = 0, \theta' = -1 \; at \; y = 0$$

$$\frac{\partial f}{\partial y} \to \frac{3}{2}\frac{\sin x}{x}, \frac{\partial^2 f}{\partial y^2} = 0, \theta \to 0 \; at \; y \to \infty \tag{14}$$

At the lower stagnation point ($x \approx 0$), then equations (12) and (13) become

$$f''' + 2ff'' - f'^2 + \frac{9}{4} + \lambda\theta + 2K(f'f''' - ff'''' - f''^2) - M\left(f' - \frac{3}{2}\right) = 0 \tag{15}$$

$$\frac{1}{P_r}\theta'' + 2f\theta' = 0 \tag{16}$$

with the boundary conditions.

$$f(0) = f'(0) = 0, \theta'(0) = -1 \; at \; y = 0$$

$$f' \to \frac{3}{2}, f'' = 0, \theta \to 0 \; at \; y \to \infty \tag{17}$$

NUMERICAL TECHNIQUE

Before discretizing (15) and (16), the finite difference method of Keller-Box is explained more details as follows.

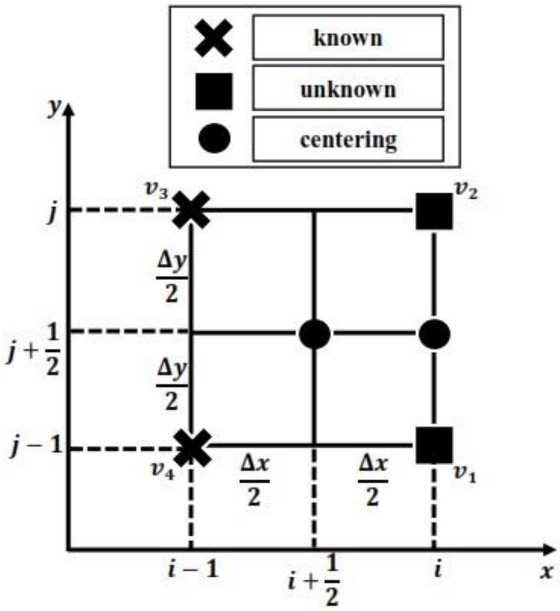

Figure 2. Two-Dimensional Keller-Box Method Stencil

Figure 2 gives illustration Keller-Box that this method has to replace the higher derivative to be first derivative and this causes the computational effort per time step expensive. This method are efficient and appropriate to be applied for solving the parabolic partial differential equation problems. Based on Figure 2, step size of Keller-Box is average between j and $j-1$ for axis y and between i

and $i-1$ for axis x where stencil consist of v_1, v_2, v_3, and v_4. Stencil v_3 and v_4 are known at boundary conditions. Meanwhile, stencil v_1 and v_2 will be obtained from the calculation of stencil v_1 and v_2 with half step size for special x and y namely $\frac{\Delta x}{2}$ and $\frac{\Delta y}{2}$ respectively as centering steps. This method has two accuracy in both space x and y where the step size of space x and y to be arbitrary, in other words in uniform or non-uniform step size. It follows from (15) and (16), in which the momentum and energy equations are in steady state, incompressible, and only at lower stagnation point ($x \approx 0$), then above finite difference method Keller-Box stencil is only dependent to the space y. Furthermore, Figure 2 can be represented in Figure 3.

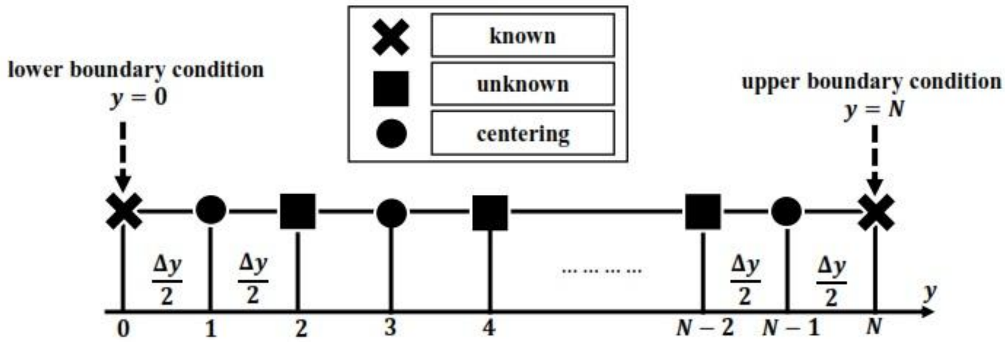

Figure 3. One-Dimensional Keller-Box Method Stencil

Figure 3 gives illustration Keller-Box method stencil when it only depends on space y with the step size of $\frac{\Delta y}{2}$, lower boundary condition $y = 0$ and upper boundary condition $y = N$. In addition, y is boundary layer thickness caused the shear stress between viscoelastic fluid and surface of solid sphere. Actually this One-Dimensional Keller-Box Method Stencil concept is identical to Figure 2 for Two-Dimensional Keller-Box Method Stencil where the step size is $\frac{\Delta y}{2}$. To facilitate in numerical processes, then (15) and (16) are transformed into first order as written in equations (18)-(23).

$$f' = u \tag{18}$$

$$u' = v \tag{19}$$

$$v' = w \tag{20}$$

$$s' = t \tag{21}$$

$$w + 2fv - u^2 + \frac{9}{4} + \lambda s + 2K(uw - fw' - v^2) - M\left(u - \frac{3}{2}\right) = 0 \qquad (22)$$

$$\frac{1}{P_r}t' + 2ft = 0 \qquad (23)$$

where s is identical to θ and the boundary conditions are written in (24).

$$f(0) = u(0) = 0, t(0) = -1 \; at \; y = 0$$

$$u \to \frac{3}{2}, v = 0, s \to 0 \; at \; y \to \infty \qquad (24)$$

Based on Figure 3, then (18) to (23) can be discretized using backward difference, then we employ the results by newton linearization technique to get (25)

$$\delta f_j - \delta f_{j-1} - \frac{h_j}{2}\delta u_j - \frac{h_j}{2}\delta u_{j-1} = (r_1)_j$$

$$\delta u_j - \delta u_{j-1} - \frac{h_j}{2}\delta v_j - \frac{h_j}{2}\delta v_{j-1} = (r_2)_j$$

$$\delta v_j - \delta v_{j-1} - \frac{h_j}{2}\delta w_j - \frac{h_j}{2}\delta w_{j-1} = (r_3)_j$$

$$\delta s_j - \delta s_{j-1} - \frac{h_j}{2}\delta t_j - \frac{h_j}{2}\delta t_{j-1} = (r_4)_j$$

$$[(a_1)_j]\delta w_j + [(a_2)_j]\delta w_{j-1} + [(a_3)_j]\delta v_j + [(a_4)_j]\delta v_{j-1} + [(a_5)_j]\delta f_j$$
$$+[(a_6)_j]\delta f_{j-1} + [(a_7)_j]\delta u_j + [(a_8)_j]\delta u_{j-1} + [(a_9)_j]\delta s_j + [(a_{10})_j]\delta s_{j-1}$$
$$= (r_5)_j [(b_1)_j]\delta t_j + [(b_2)_j]\delta t_{j-1} + [(b_3)_j]\delta f_j + [(b_4)_j]\delta f_{j-1} = (r_6)_j \qquad (25)$$

Where

$$(r_1)_j = f_{j-1} - f_j + \frac{h_j}{2}(u_j + u_{j-1})$$

$$(r_2)_j = u_{j-1} - u_j + \frac{h_j}{2}(v_j + v_{j-1})$$

$$(r_3)_j = v_{j-1} - v_j + \frac{h_j}{2}(w_j + w_{j-1})$$

$$(r_4)_j = s_{j-1} - s_j + \frac{h_j}{2}(t_j + t_{j-1})$$

$$(r_5)_j = -h_j w_{j-\frac{1}{2}} - 2h_j f_{j-\frac{1}{2}} v_{j-\frac{1}{2}} + h_j u^2_{j-\frac{1}{2}} - \frac{9}{4}h_j - 2h_j u_{j-\frac{1}{2}} w_{j-\frac{1}{2}}$$
$$+2K f_{j-\frac{1}{2}}(w_j - w_{j-1}) + 2K h_j v^2_{j-\frac{1}{2}} - \lambda h_j s_{j-\frac{1}{2}} + h_j M\left(u_{j-\frac{1}{2}} - \frac{3}{2}\right)$$

$$(r_6)_j = \frac{1}{P_r}(t_{j-1} - t_j) - 2h_j f_{j-\frac{1}{2}} t_{j-\frac{1}{2}}$$

$$\left[(a_1)_j\right] = \frac{h_j}{2} + Kh_j u_{j-\frac{1}{2}} - 2Kf_{j-\frac{1}{2}}; \quad \left[(a_2)_j\right] = \left[(a_1)_j\right]$$

$$\left[(a_3)_j\right] = h_j f_{j-\frac{1}{2}} - 2Kh_j v_{j-\frac{1}{2}}; \ \left[(a_4)_j\right] = \left[(a_3)_j\right]$$

$$\left[(a_5)_j\right] = h_j v_{j-\frac{1}{2}} - K\left(w_j - w_{j-1}\right); \ \left[(a_6)_j\right] = \left[(a_5)_j\right]$$

$$\left[(a_7)_j\right] = -h_j u_{j-\frac{1}{2}} - \frac{h_j}{2} M + Kh_j w_{j-\frac{1}{2}}; \ \left[(a_8)_j\right] = \left[(a_7)_j\right]$$

$$\left[(a_9)_j\right] = \left(\frac{\lambda h_j}{2}\right); \ \left[(a_{10})_j\right] = \left[(a_9)_j\right]$$

$$\left[(b_1)_j\right] = \frac{1}{P_r} + h_j f_{j-\frac{1}{2}}; \ \left[(b_2)_j\right] = -\frac{1}{P_r} + h_j f_{j-\frac{1}{2}}$$

$$\left[(b_3)_j\right] = h_j t_{j-\frac{1}{2}}; \ \left[(b_4)_j\right] = \left[(b_3)_j\right]$$

TRIDIAGONAL BLOCK MATRIX

It follows from (25), then it can be iterated for $i = 1,2,\dots,N$ with the boundary conditions $\delta f_0 = 0$, $\delta u_0 = 0, \delta s_0 = 0, \delta u_N = 0, \delta v_N = 0$, and $\delta s_N = 0$. Then, obtained the block matrix as follows.

- Iteration 1

$$\delta f_1 - \frac{h_1}{2}\delta u_1 = (r_1)_1$$

$$\delta u_1 - \frac{h_1}{2}\delta v_1 - \frac{h_1}{2}\delta v_0 = (r_2)_1$$

$$\delta v_1 - \delta v_0 - \frac{h_1}{2}\delta w_1 - \frac{h_1}{2}\delta w_0 = (r_3)_1$$

$$\delta s_1 - \frac{h_1}{2}\delta t_1 - \frac{h_1}{2}\delta t_0 = (r_4)_1$$

$$[(a_1)_1]\delta w_1 + [(a_2)_1]\delta w_0 + [(a_3)_1]\delta v_1 + [(a_4)_1]\delta v_0 + [(a_5)_1]\delta f_1$$
$$+[(a_7)_1]\delta u_1 + [(a_9)_1]\delta s_1 = (r_5)_1$$

$$[(b_1)_1]\delta t_1 + [(b_2)_1]\delta t_0 + [(b_3)_1]\delta f_1 = (r_6)_1$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{h_1}{2} & 0 & 0 & 0 & 0 & 0 \\ -1 & -\frac{h_1}{2} & 0 & 0 & -\frac{h_1}{2} & 0 \\ 0 & 0 & -\frac{h_1}{2} & 0 & 0 & -\frac{h_1}{2} \\ [(a_4)_1] & [(a_2)_1] & 0 & [(a_5)_1] & [(a_1)_1] & 0 \\ 0 & 0 & [(b_2)_1] & [(b_3)_1] & 0 & [(b_1)_1] \end{bmatrix} \begin{bmatrix} \delta v_0 \\ \delta w_0 \\ \delta t_0 \\ \delta f_1 \\ \delta w_1 \\ \delta t_1 \end{bmatrix} +$$

$$\begin{bmatrix} -\frac{h_1}{2} & 0 & 0 & 0 & 0 & 0 \\ 1 & -\frac{h_1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ [(a_7)_1] & [(a_3)_1] & [(a_9)_1] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta v_1 \\ \delta s_1 \\ \delta f_2 \\ \delta w_2 \\ \delta t_2 \end{bmatrix} = \begin{bmatrix} (r_1)_1 \\ (r_2)_1 \\ (r_3)_1 \\ (r_4)_1 \\ (r_5)_1 \\ (r_6)_1 \end{bmatrix}$$

- Iteration 2

$$\delta f_2 - \delta f_1 - \frac{h_2}{2}\delta u_2 - \frac{h_2}{2}\delta u_1 = (r_1)_2$$

$$\delta u_2 - \delta u_1 - \frac{h_2}{2}\delta v_2 - \frac{h_2}{2}\delta v_1 = (r_2)_2$$

$$\delta v_2 - \delta v_1 - \frac{h_2}{2}\delta w_2 - \frac{h_2}{2}\delta w_1 = (r_3)_2$$

$$\delta s_2 - \delta s_1 - \frac{h_2}{2}\delta t_2 - \frac{h_2}{2}\delta t_1 = (r_4)_2$$

$$[(a_1)_2]\delta w_2 + [(a_2)_2]\delta w_1 + [(a_3)_2]\delta v_2 + [(a_4)_2]\delta v_1 + [(a_5)_2]\delta f_2$$

$$+[(a_6)_2]\delta f_1 + [(a_7)_2]\delta u_2 + [(a_8)_2]\delta u_1 + [(a_9)_2]\delta s_2$$

$$+[(a_{10})_2]\delta s_1 = (r_5)_2$$

$$[(b_1)_2]\delta t_2 + [(b_2)_2]\delta t_1 + [(b_3)_2]\delta f_2 + [(b_4)_2]\delta f_1 = (r_6)_2$$

$$\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{h_2}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{h_2}{2} \\ 0 & 0 & 0 & [(a_6)_2] & [(a_2)_2] & 0 \\ 0 & 0 & 0 & [(b_4)_2] & 0 & [(b_2)_2] \end{bmatrix} \begin{bmatrix} \delta u_0 \\ \delta v_0 \\ \delta s_0 \\ \delta f_1 \\ \delta w_1 \\ \delta t_1 \end{bmatrix} +$$

$$\begin{bmatrix} -\frac{h_2}{2} & 0 & 0 & 1 & 0 & 0 \\ -1 & -\frac{h_2}{2} & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -\frac{h_2}{2} & 0 \\ 0 & 0 & -1 & 0 & 0 & -\frac{h_2}{2} \\ [(a_8)_2] & [(a_4)_2] & [(a_{10})_2] & [(a_5)_2] & [(a_1)_2] & 0 \\ 0 & 0 & 0 & [(b_3)_2] & 0 & [(b_1)_2] \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta v_1 \\ \delta s_1 \\ \delta f_2 \\ \delta w_2 \\ \delta t_2 \end{bmatrix} +$$

$$\begin{bmatrix} -\frac{h_2}{2} & 0 & 0 & 0 & 0 & 0 \\ 1 & -\frac{h_2}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ [(a_7)_2] & [(a_3)_2] & [(a_9)_2] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta u_2 \\ \delta v_2 \\ \delta s_2 \\ \delta f_3 \\ \delta w_3 \\ \delta t_3 \end{bmatrix} = \begin{bmatrix} (r_1)_2 \\ (r_2)_2 \\ (r_3)_2 \\ (r_4)_2 \\ (r_5)_2 \\ (r_6)_2 \end{bmatrix}$$

- Iteration N

$$\delta f_N - \delta f_{N-1} - \frac{h_N}{2}\delta u_{N-1} = (r_1)_N$$

$$-\delta u_{N-1} - \frac{h_N}{2}\delta v_{N-1} = (r_2)_N$$

$$-\delta v_{N-1} - \frac{h_N}{2}\delta w_N - \frac{h_N}{2}\delta w_{N-1} = (r_3)_N$$

$$-\delta s_{N-1} - \frac{h_N}{2}\delta t_N - \frac{h_N}{2}\delta t_{N-1} = (r_4)_N$$

$$[(a_1)_N]\delta w_N + [(a_2)_N]\delta w_{N-1} + [(a_4)_N]\delta v_{N-1} + [(a_5)_N]\delta f_N$$

$$+[(a_6)_N]\delta f_{N-1} + [(a_7)_N]\delta u_N + [(a_8)_N]\delta u_{N-1} + [(a_{10})_N]\delta s_{N-1} = (r_5)_N$$

$$[(b_1)_N]\delta t_N + [(b_2)_N]\delta t_{N-1} + [(b_3)_N]\delta f_N + [(b_4)_N]\delta f_{N-1} = (r_6)_N$$

$$\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{h_N}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{h_N}{2} \\ 0 & 0 & 0 & [(a_6)_N] & [(a_2)_N] & 0 \\ 0 & 0 & 0 & [(b_4)_N] & 0 & [(b_2)_N] \end{bmatrix} \begin{bmatrix} \delta u_{N-2} \\ \delta v_{N-2} \\ \delta s_{N-2} \\ \delta f_{N-1} \\ \delta w_{N-1} \\ \delta t_{N-1} \end{bmatrix} +$$

$$\begin{bmatrix} -\frac{h_N}{2} & 0 & 0 & 1 & 0 & 0 \\ -1 & -\frac{h_N}{2} & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -\frac{h_N}{2} & 0 \\ 0 & 0 & -1 & 0 & 0 & -\frac{h_N}{2} \\ [(a_8)_N] & [(a_4)_N] & [(a_{10})_N] & [(a_5)_N] & [(a_1)_N] & 0 \\ 0 & 0 & 0 & [(b_3)_N] & 0 & [(b_1)_N] \end{bmatrix} \begin{bmatrix} \delta u_{N-1} \\ \delta v_{N-1} \\ \delta s_{N-1} \\ \delta f_N \\ \delta w_N \\ \delta t_N \end{bmatrix} = \begin{bmatrix} (r_1)_N \\ (r_2)_N \\ (r_3)_N \\ (r_4)_N \\ (r_5)_N \\ (r_6)_N \end{bmatrix}$$

These results can be rewritten as follows

for $j = 1$: $[A_1][\delta_1] + [C_1][\delta_2] = [r_1]$

for $j = 2$: $[B_2][\delta_1] + [A_2][\delta_2] + [C_2][\delta_3] = [r_2]$

for $j = 3$: $[B_3][\delta_2] + [A_3][\delta_3] + [C_3][\delta_4] = [r_3]$

$\vdots$

for $j = N-1$: $[B_{N-1}][\delta_{N-2}] + [A_{N-1}][\delta_{N-1}] + [C_{N-1}][\delta_N] = [r_{N-1}]$

for $j = N$: $[B_N][\delta_{N-1}] + [A_N][\delta_N] = [r_N]$ (26)

where

$$[A_1] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{h_1}{2} & 0 & 0 & 0 & 0 & 0 \\ -1 & -\frac{h_1}{2} & 0 & 0 & -\frac{h_1}{2} & 0 \\ 0 & 0 & -\frac{h_1}{2} & 0 & 0 & -\frac{h_1}{2} \\ [(a_4)_1] & [(a_2)_1] & 0 & [(a_5)_1] & [(a_1)_1] & 0 \\ 0 & 0 & [(b_2)_1] & [(b_3)_1] & 0 & [(b_1)_1] \end{bmatrix}$$

$$[A_j] = \begin{bmatrix} -\frac{h_2}{2} & 0 & 0 & 1 & 0 & 0 \\ -1 & -\frac{h_2}{2} & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -\frac{h_2}{2} & 0 \\ 0 & 0 & -1 & 0 & 0 & -\frac{h_2}{2} \\ [(a_8)_2] & [(a_4)_2] & [(a_{10})_2] & [(a_5)_2] & [(a_1)_2] & 0 \\ 0 & 0 & 0 & [(b_3)_2] & 0 & [(b_1)_2] \end{bmatrix}, \quad 2 \le j \le N$$

$$[B_j] = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{h_2}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{h_2}{2} \\ 0 & 0 & 0 & [(a_6)_2] & [(a_2)_2] & 0 \\ 0 & 0 & 0 & [(b_4)_2] & 0 & [(b_2)_2] \end{bmatrix}, \quad 2 \le j \le N$$

$$[C_j] = \begin{bmatrix} -\frac{h_2}{2} & 0 & 0 & 0 & 0 & 0 \\ 1 & -\frac{h_2}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ [(a_7)_2] & [(a_3)_2] & [(a_9)_2] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad 1 \le j \le N-1$$

$$[\delta_1] = \begin{bmatrix} \delta v_0 \\ \delta w_0 \\ \delta t_0 \\ \delta f_1 \\ \delta w_1 \\ \delta t_1 \end{bmatrix}, \qquad [\delta_j] = \begin{bmatrix} \delta u_{j-1} \\ \delta v_{j-1} \\ \delta s_{j-1} \\ \delta f_j \\ \delta w_j \\ \delta t_j \end{bmatrix}, \quad 2 \le j \le N$$

$$[r_j] = \begin{bmatrix} (r_1)_j \\ (r_2)_j \\ (r_3)_j \\ (r_4)_j \\ (r_5)_j \\ (r_6)_j \end{bmatrix}, \qquad 1 \le j \le N$$

In other words, (26) can be expressed in vector matrix form.

$$\boldsymbol{A\delta} = \boldsymbol{r} \tag{27}$$

where

$$\boldsymbol{A} = \begin{bmatrix} [A_1] & [C_1] & 0 & 0 & \dots & 0 \\ [B_2] & [A_2] & [C_2] & 0 & \dots & 0 \\ 0 & [B_3] & [A_3] & [C_3] & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & [B_{N-1}] & [A_{N-1}] & [C_{N-1}] \\ 0 & 0 & \dots & 0 & [B_N] & [A_N] \end{bmatrix}$$

$$\boldsymbol{\delta} = \begin{bmatrix} [\delta_1] \\ [\delta_2] \\ [\delta_3] \\ \vdots \\ [\delta_{N-1}] \\ [\delta_N] \end{bmatrix}, \boldsymbol{r} = \begin{bmatrix} [r_1] \\ [r_2] \\ [r_3] \\ \vdots \\ [r_{N-1}] \\ [r_N] \end{bmatrix}$$

To solve this block matrix, then Decomposition LU method is used to make easy in factorization of the form

$$\boldsymbol{A} = \boldsymbol{LU} \tag{28}$$

where

$$\boldsymbol{L} = \begin{bmatrix} [\alpha_1] & 0 & 0 & 0 & \dots & 0 \\ [B_2] & [\alpha_2] & 0 & 0 & \dots & 0 \\ 0 & [B_3] & [\alpha_3] & 0 & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & [B_{N-1}] & [\alpha_{N-1}] & 0 \\ 0 & 0 & \dots & 0 & [B_N] & [\alpha_N] \end{bmatrix}$$

and

$$\boldsymbol{U} = \begin{bmatrix} [I] & [\Gamma_1] & 0 & 0 & \dots & 0 \\ 0 & [I] & [\Gamma_2] & 0 & \dots & 0 \\ 0 & 0 & [I] & [\Gamma_3] & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & [I] & [\Gamma_{N-1}] \\ 0 & 0 & \dots & 0 & 0 & [I] \end{bmatrix}$$

To compute matrix $[\alpha_j]$ and $[\Gamma_j]$, then the following ones are defined as

$[\alpha_1] = [A_1]$

$[\alpha_j] = [A_j] - [B_j][\Gamma_{j-1}], \quad j = 2,3,\dots,N$

$[\alpha_j][\Gamma_j] = [C_j], \quad j = 1,2,\dots,N-1$

Further, by substituting (28) into (27), then obtained

$$\boldsymbol{LU\delta} = \boldsymbol{r} \tag{29}$$

Next, by defining $\boldsymbol{U\delta} = \boldsymbol{w}$, (29) can be expressed as

$$\boldsymbol{Lw} = \boldsymbol{r} \tag{30}$$

where

$$\boldsymbol{w} = \begin{bmatrix} [w_1] \\ [w_2] \\ [w_3] \\ \vdots \\ [w_{N-1}] \\ [w_N] \end{bmatrix}$$

To solve (30), then the following ones are defined as

$[\alpha_1][w_1] = [r_1]$

$[\alpha_j][w_j] = [r_j] - [B_j][w_{j-1}], \quad j = 2,3,\dots,N$

and $[w_j]$ is computed by forward sweep technique. Further, by obtaining $[w_j]$ this can be used to obtain the solution $[\delta_j]$ using backward sweep technique as shown below

$$[\delta_j] = [w_j]$$

$$[\delta_j] = [w_j] - [\Gamma_j][\delta_{j+1}], \quad j = 1,2,\dots,N-1$$

The iteration is continuing until converging criterion is fulfilled. Based on Cebeci and Bradshaw (1977), $v(x,0,t)$ is converging criterion, so the iteration will be stopped when $|\delta v(x,0,t)| < \epsilon$ where ϵ is very small value.

NUMERICAL RESULTS

The numerical results of this research are the effect of pandtl number (P_r), viscoelastic (K), mixed convection (λ), and MHD (M) parameter to temperature profile (θ) and velocity profile (f').

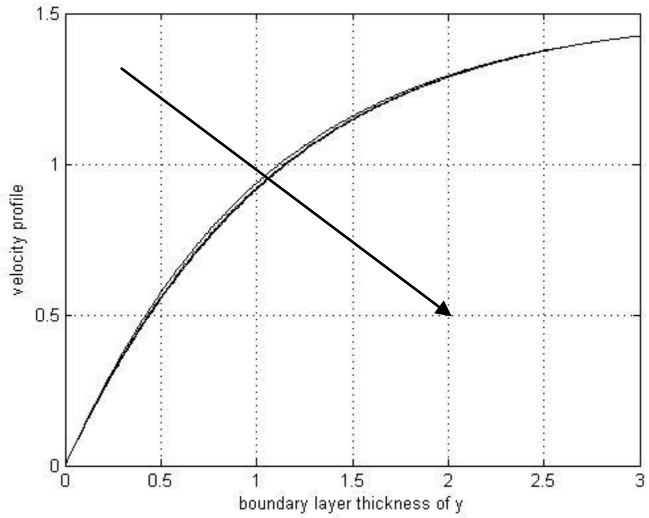

Figure 4. Velocity Profile (f') with Prandtl Number ($P_r = 1,5,10,15$) and Boundary Layer Thickness y

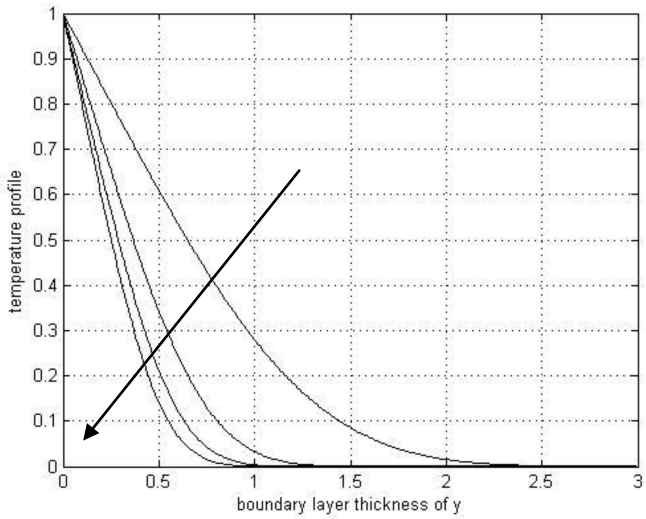

Figure 5. Temperature Profile (θ) with Prandtl Number ($P_r = 1,5,10,15$) and Boundary Layer Thickness y

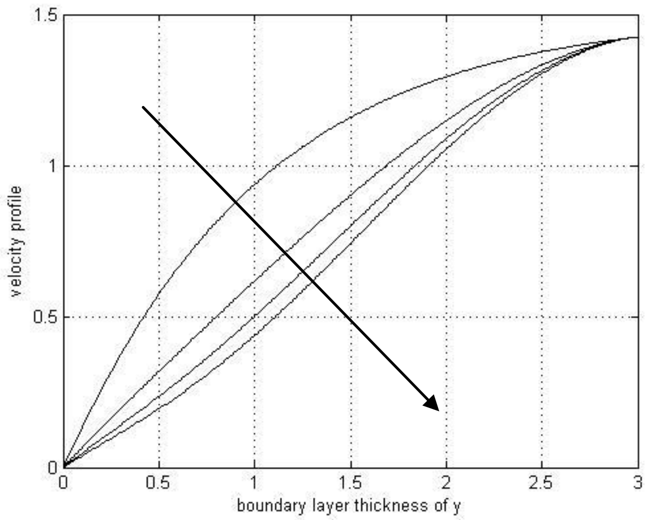

Figure 6. Velocity Profile (f') with Viscoelastic Parameter ($K = 1,5,10,15$) and Boundary Layer Thickness y

In Figure 4 it can be seen that as the Prandtl number increases, the fluid flow velocity decreases at every certain point y. The influence of the Prandtl number on the fluid temperature decreases at every certain point y. Based on the nature of the convection flow that moves from low density to high density, the low temperature has a high density, resulting in a smaller velocity profile.

Figure 5 shows that the higher the Prandtl number, the smaller the temperature of the fluid passing through the spherical surface at any given point y, this is because the Prandtl number is related to the distribution of heat, so the heat distribution increases which results in the fluid temperature getting smaller at every certain point y.

Figure 6 shows that as the viscoelastic parameter increases, the viscosity level also increases, this causes the friction value between the fluid and the spherical surface media to increase, so this affects the fluid flow velocity which is getting smaller at every certain point y.

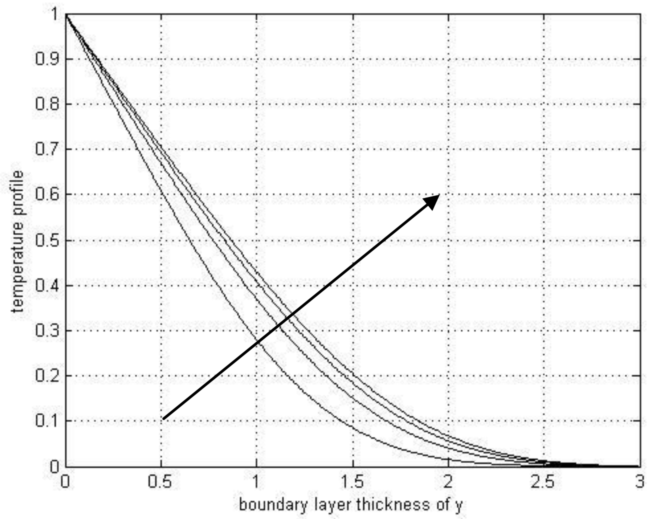

Figure 7. Temperature Profile (θ) with Viscoelastic Parameter ($K = 1,5,10,15$) and Boundary Layer Thickness y

Figure 7 shows that the increase in the viscoelastic parameter causes the fluid temperature to increase at every certain point y, this is because the influence of the increasing value of the viscoelastic parameter causes the frictional force between the viscoelastic fluid and the surface of a sphere to also increase. This results in the heat generated from the friction also getting bigger, so that the temperature of the fluid passing through the surface of the sphere is getting bigger at any given point y.

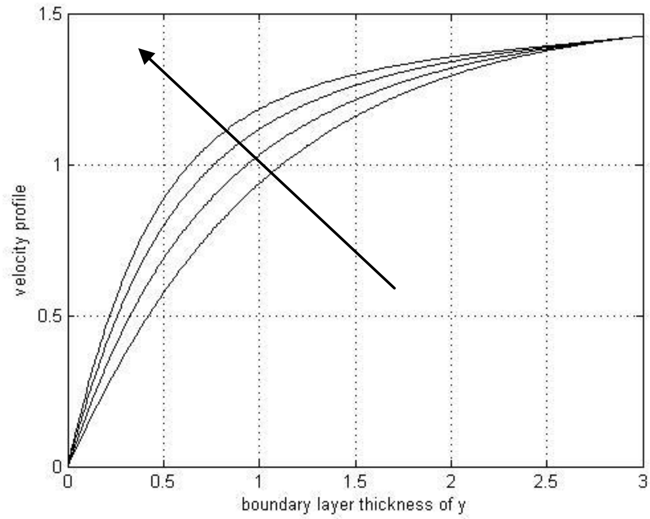

Figure 8. Velocity Profile (f') with Mixed Convection Parameter ($\lambda = 1{,}5{,}10{,}15$) and Boundary Layer Thickness y

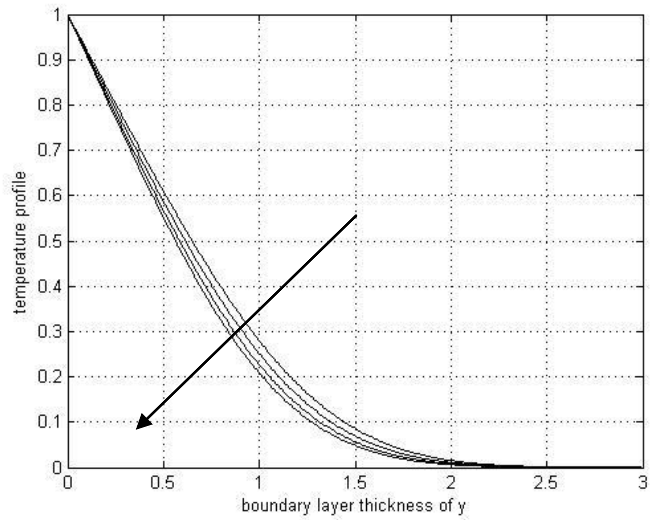

Figure 9. Temperature Profile (θ) with Mixed Convection Parameter ($\lambda = 1{,}5{,}10{,}15$) and Boundary Layer Thickness y

Figure 8 shows that the higher the convection parameter of the mixture, the greater the fluid flow velocity at every certain point y, this is due to the influence of external forces, causing the fluid convection flow velocity to increase at any given point y. Meanwhile, Figure 9 shows that the increase in the convection parameters of the mixture, the greater the temperature at every certain point y, this is because the fluid flow velocity that increases at every certain point

y indicates that the temperature of the viscoelastic fluid passing through the surface of a sphere is getting bigger at every certain point y. So that the buoyancy forces that are getting bigger cause the fluid temperature to get bigger at every certain point y.

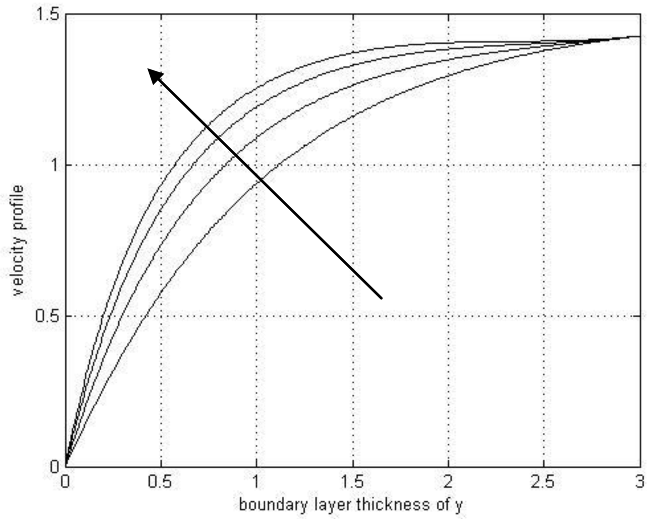

Figure 10. Velocity Profile (f') with MHD Parameter ($M = 1,5,10,15$) and Boundary Layer Thickness y

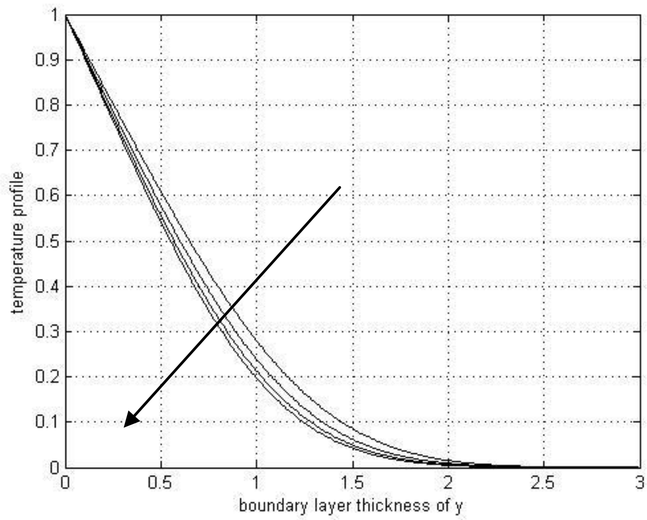

Figure 11. Temperature Profile (θ) with MHD Parameter ($M = 1,5,10,15$) and Boundary Layer Thickness y

Figure 10 shows that the higher the hydrodynamic parameters of the magnet, the greater the fluid flow velocity at any given point y, this is due to the decreasing effect of the density of the viscoelastic fluid as shown by the correlation. The decreasing density causes the buoyant force to work to be greater, thus causing the velocity to increase at any given point y. Moreover, Figure 11 shows that the higher the hydrodynamic parameters of the magnet, the higher the temperature at any given point y, this is because a correlation between the hydrodynamic parameters of the magnet is inversely proportional to the density of the viscoelastic fluid. This causes the hydrodynamic parameters of the magnet to increase, the density decreases, so that the temperature increases at every certain point y.

CONCLUSIONS

In this research, the problem of mixed convection flow on viscoelastic fluid over a sphere surface with the effect of MHD is studied. The non-similar equations of momentum and energy are solved numerically using the finite difference method with the iterative method. The effect of Prantdl number, viscoelastic, mixed convection, and MHD parameter to the characteristic of temperature profile (θ) and velocity profile (f') have been obtained and discussed. Then, the conclusions of this research can be written as follows.

1. The mathematical model of this research is obtained from the derivation of Navier-Stokes including continuity, momentum, and energy equations. These equations are transformed into non-dimensional equations. Further, the non-dimensional equations are transformed into non-similar equations as written below.

$$f''' + 2ff'' - f'^2 + \frac{9}{4} + \lambda\theta + 2K(f'f''' - ff'''' - f''^2) - M\left(f' - \frac{3}{2}\right) = 0$$

and

$$\frac{1}{P_r}\theta'' + 2f\theta' = 0$$

2. The non-similar equations are solved numerically using Keller-Box method by transforming the higher order into first order. This discretization results are solved using newton linearization technique and to find factorization of solution using decomposition LU where converging criterion is $|\delta v(x,0,t)| < \epsilon$.

3. The numerical results are given in Figure 4 to Figure 11 with the effect of Prantdl number, viscoelastic, mixed convection, and MHD parameter to the characteristic of temperature profile (θ) and velocity profile (f')

APPENDIX

In this section, we present the technique to get the radial distance of sphere$\bar{r}(\overline{x})$ in equation (1). We only focus on some variables and parameters, such as diameter of circle of a, radial distance of $\bar{r}$, $\bar{x}$-axis (along a surface of circle), and angle of $\bar{x}/a$ as shown Figure 12.

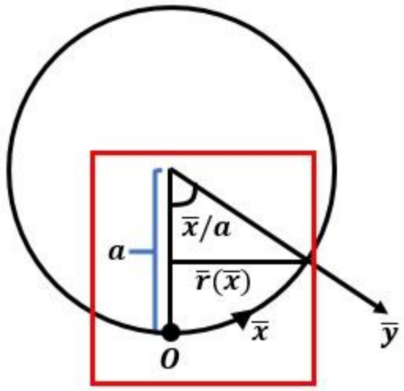

Figure 12. Geometry of Physical Model

Figure 12 is the geometry of main problem that we want to write here and when we focus on red area in Figure 12, we will find a geometry of triangle. Furthermore, we will ignore $\bar{x}$-axis as shown in Figure 13.

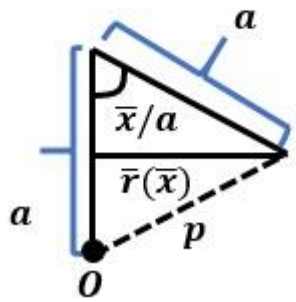

Figure 13. Geometry of Physical Model Without Arc of Circle $\bar{x}$

According to Figure 13, we can find the radial distance of $\bar{r}$ by finding the value of p firstly. To do that, we can use the formula of triangle.

$$p^2 = s_1^2 + s_2^2 - 2s_1 s_2 \cos\theta$$

In this case, s_1 and s_2 are sides of triangle, where $s_1 = s_2 = a$. Furthermore, we obtain

$$\begin{aligned} p^2 &= a^2 + a^2 - 2a^2 \cos\theta \\ &= 2a^2 - 2a^2 \cos\theta \\ &= 2a^2(1 - \cos\theta) \end{aligned}$$

Since $\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right) \Leftrightarrow 1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$, then obtained

$$\begin{aligned} p^2 &= 2a^2\left(2\sin^2\left(\frac{\theta}{2}\right)\right) \\ &= 4a^2 \sin^2\left(\frac{\theta}{2}\right) \end{aligned}$$

which gives

$$p = \pm 2a \sin\left(\frac{\theta}{2}\right)$$

The last step is to find the radial distance of $\bar{r}$ by using the Pythagoras formula of two triangles as shown in Figure 14.

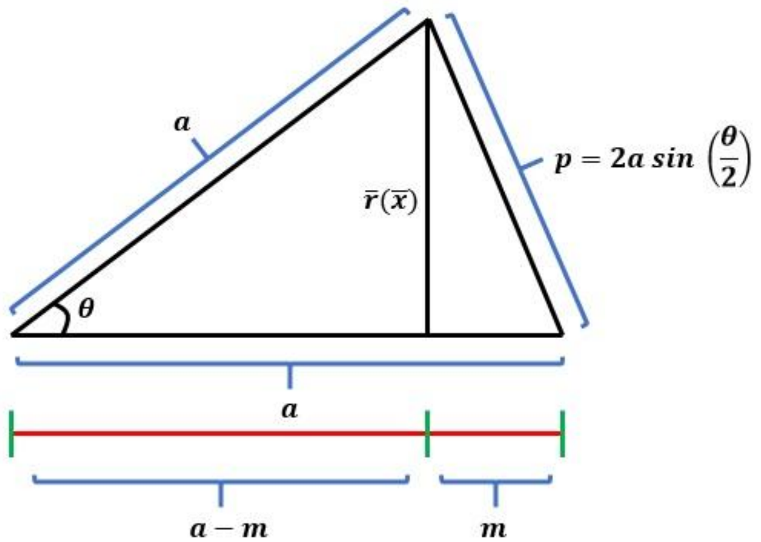

Figure 14. Geometry of physical model in a triangle

which gives the formula

$$a^2 - (a-m)^2 = \left(2a\sin\left(\frac{\theta}{2}\right)\right)^2 - m^2$$

$$\Leftrightarrow a^2 - (a^2 - 2am + m^2) = 4a^2\sin^2\left(\frac{\theta}{2}\right) - m^2$$

$$\Leftrightarrow 2am - m^2 = 4a^2\sin^2\left(\frac{\theta}{2}\right) - m^2$$

$$\Leftrightarrow 2am = 4a^2\sin^2\left(\frac{\theta}{2}\right)$$

$$\Leftrightarrow m = 2a\sin^2\left(\frac{\theta}{2}\right)$$

The next step is to substitute the value of m into the following formula.

$$\bar{r} = \sqrt{p^2 - m^2}$$

Then, one has

$$\bar{r} = \sqrt{\left(2a\sin\left(\frac{\theta}{2}\right)\right)^2 - \left(2a\sin^2\left(\frac{\theta}{2}\right)\right)^2}$$

$$= \sqrt{4a^2\sin^2\left(\frac{\theta}{2}\right) - 4a^2\sin^4\left(\frac{\theta}{2}\right)}$$

$$= \sqrt{4a^2\sin^2\left(\frac{\theta}{2}\right)\left(1 - \sin^2\left(\frac{\theta}{2}\right)\right)}$$

$$= \sqrt{4a^2\, sin^2\left(\frac{\theta}{2}\right) cos^2\left(\frac{\theta}{2}\right)}$$

$$= 2a\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$= a\left(2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)$$

$$= a\sin\theta$$

Finally, we assume $\theta = \bar{x}/a$, then we obtain the desired result of radial distance $\bar{r} = a\, sin\, (\bar{x}/a)$.

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